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1989 J. Phys. A: Math. Gen. 22 4263

(http://iopscience.iop.org/0305-4470/22/19/019)

J. Phys. A: Math. Gen. **22** (1989) 4263-4269. Printed in the UK

COMMENT

Rigorous result for the area-weighted moments of convex polygons on the square lattice

K Y Lin

Physics Department, National Tsing Hua University, Hsinchu, Taiwan 30043, Republic of China

Received 23 February 1989, in final form 15 May 1989

Abstract. A generating function for the area-weighted moments of the number of convex polygons on the square lattice is derived rigorously. A special case of our result was obtained first by Enting and Guttmann non-rigorously. Their calculation is based on series expansions.

A convex polygon on the square lattice is a self-avoiding polygon whose number of steps is equal to the perimeter of the minimum bounding rectangle. Any straight line on the bonds of the dual lattice cuts the bonds of the convex polygon at most twice. Recently Guttmann and Enting (1988) enumerated the number P_n of convex polygons with n steps up to $n=64$. They determined the precise recurrence relation for P_n by systematic searching and then obtained the generating function

$$P(x)=\sum_{n=2}^{\infty} P_{2n}x^{2n}=x^4(1-6x^2+11x^4-4x^6)/(1-4x^2)^2-4x^8/(1-4x^2)^{3/2}. \tag{1}$$

Unknown to Guttmann and Enting, this result was first derived by Delest and Viennot (1984), who also proved that

$$P_{2m+8}=(2m+11)2^{2m}-4(2m+1)(2m)!/(m!)^2. \tag{2}$$

Lin and Chang (1988) generalised this result to

$$\begin{aligned} P(x,y)&=\sum_{r,s=1}^{\infty} P_{r,s}y^{2r}x^{2s} \\ &=x^2y^2[1-3x^2-3y^2+3x^4+3y^4+5x^2y^2-x^6-y^6 \\ &\quad -x^2y^4-x^4y^2-x^2y^2(x^2-y^2)^2]/\Delta^2-4x^4y^4/\Delta^{3/2} \end{aligned} \tag{3}$$

where

$$\Delta=1-2x^2-2y^2+(x^2-y^2)^2 \tag{4}$$

and $P_{r,s}$ is the number of convex polygons with vertical height r and horizontal width s. The authors prove this formula by deriving recurrence relation for convex polygons with a given top-row width. A similar method was used by Temperley (1956) in some related problems. When $x=y$, equation (3) reduces to (1). Another proof of (1) was given by Kim (1988).

Recently Enting and Guttmann (1989) considered two generating functions for the area-weighted moments of the number of convex polygons:

$$P_1(x)=\sum_n x^n\left(\sum_k kc_{n,k}\right) \qquad P_2(x)=\sum_n x^n\sum_k [k(k-1)/2]c_{n,k} \tag{5}$$

where $c_{n,k}$ is the number of convex polygons with n steps and area k. Based on the series expansions, they obtained non-rigorously

$$P_1(x)=x^4[(1-12x^2+50x^4-76x^6+42x^8-48x^{10}+32x^{12})/(1-4x^2)^4 + 4x^4/(1-4x^2)^{5/2}] \tag{6}$$

$$P_2(x)=x^6[R(x)/(1-4x^2)^6+S(x)/(1-4x^2)^{9/2}] \tag{7}$$

where

$$R(x)=2+5x^2-224x^4+1306x^6-3352x^8+4536x^{10} -3424x^{12}+1664x^{14}-512x^{16} \tag{8}$$
$$S(x)=-29x^2+172x^4-356x^6+312x^8-120x^{10}.$$

We shall demonstrate that the method of Lin and Chang (1988) can be used to derive rigorously the following generating function:

$$P_1(x,y)=\sum_{r,s=1}^{\infty} y^{2r}x^{2s}\left(\sum_k kP_{r,s,k}\right) \tag{9}$$

where $P_{r,s,k}$ is the number of convex polygons with area k, height r and width s. When $x=y$, equation (9) reduces to (6).

We follow closely the method and notation of Lin and Chang (1988). Consider first a special case of the convex polygons. The width at the top of such a polygon is equal to the width of the minimal bounding rectangle as shown in figure 1. The generating function for the number $G_{r,s}$ of such polygons with height r and width s is

$$G(x,y)=\sum_{r,s} G_{r,s}y^{2r}x^{2s}=\sum_m G_m(x,y) \tag{10}$$

where G_m is the generating function corresponding to all polygons whose width at the top is m. It is shown by Lin and Chang (1988) that

$$\begin{aligned} G_{m+1}&=x^2G_m+y^2\sum_{n=1}^{m+1} G_n x^{2(m+1-n)} \\ G_m&=(y^2/2)(z_+^m+z_-^m) \\ G(x,y)&=x^2y^2(1-x^2)/[(1-x^2)^2-y^2] \end{aligned} \tag{11}$$

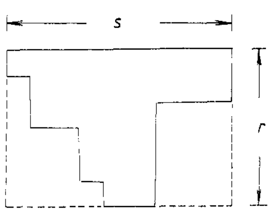

Figure 1. A convex polygon whose width at the top row equals the width of the minimal bounding rectangle.

where

$$z_{\pm} = x^2/(1 \pm y) \tag{12}$$

are the roots of

$$f(z) = (1-y^2)z^2 - 2x^2z + x^4 = 0. \tag{13}$$

Similarly we define

$$G'(x, y) = \sum_{r,s,k} kG_{r,s,k}y^{2r}x^{2s} = \sum_m G'_m(x, y) \tag{14}$$

where $G_{r,s,k}$ is the number of polygons with height r, width s and area k. We have

$$G'_1 = x^2y^2/(1-y^2)^2 \tag{15}$$

$$G'_2 = 2x^4y^2(1+2y^2)/(1-y^2)^3 \tag{16}$$

$$G'_{m+1} = x^2(G'_m + G_m) + y^2 \sum_{n=1}^{m+1} [G'_n + (m+1)G_n]x^{2(m+1-n)}. \tag{17}$$

Equation (17) is derived as follows. The polygons whose width at the top row is $m+1$ consist of two types of polygons as shown in figure 2. The first type corresponds to adding one unit square to all polygons whose top row width is m. The second type corresponds to adding one row of area $m+1$ on top of other polygons. Since each polygon with area k is counted k times, we must add correction terms in the right-hand side of (17). It follows from (17) that we have the recursion relation

$$(1-y^2)G'_{m+2} - 2x^2G'_{m+1} + x^4G'_m = (my^2+2)G_{m+2} - 2x^2G_{m+1}. \tag{18}$$

The general solution of (18) is

$$G'_m = [a_+m(m-1)/2 + b_+m + c_+]z_+^m + [a_-m(m-1)/2 + b_-m + c_-]z_-^m \tag{19}$$

where

$$a_{\pm} = -y^3/4(y \pm 1) \qquad b_{\pm} = y^2(3 \pm y)/8(1 \pm y). \tag{20}$$

The coefficients $c_{\pm}$ are determined from G'_1 and G'_2. The results are

$$c_{\pm} = -(\pm y)/8. \tag{21}$$

Summing over G'_m, we get

$$G'(x, y) = \sum_m G'_m = x^2y^2(1-x^2)^2[(1-x^2)^2 - y^2 + 2y^2x^2]/[(1-x^2)^2 - y^2]^3. \tag{22}$$

Consider next the second special case of the convex polygons as shown in figure 3. The top right-hand corner of the minimal bounding rectangle is also a corner of the polygon. The generating function for the number of polygons is

$$H(x, y) = \sum_{r,s} H_{r,s}y^{2r}x^{2s} = \sum H_m. \tag{23}$$

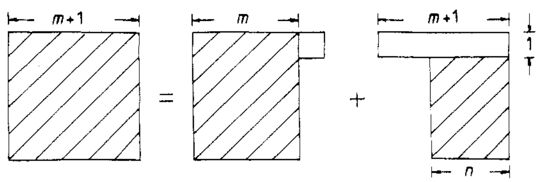

Figure 2. Graphical representation of equation (17).

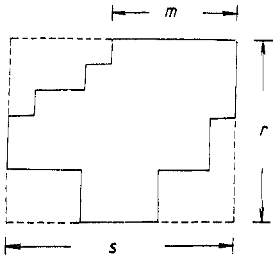

Figure 3. A convex polygon whose top right-hand corner is also the corner of the minimal bounding rectangle.

It is shown by Lin and Chang that

$$H_{m+1}=x^2H_m+y^2\sum_{n=0}^{\infty}H_{m+n+1}+y^2\sum_{n=1}^{m}G_nx^{2(m+1-n)}$$

$$H_{m+1}=d_+z_+^m+d_-z_-^m+fw^m \tag{24}$$

$$H=x^2y^2/\Delta^{1/2}$$

where Δ is defined by (4),

$$d_\pm=x^2y^2(1\pm y-x^2)/2(1\pm 2y+y^2-x^2) \qquad f=-2x^2y^4w/\Delta \tag{25}$$

and

$$w=(1+x^2-y^2-\Delta^{1/2})/2 \tag{26}$$

is a root of the equation

$$g(w)=w^2-(1+x^2-y^2)w+x^2=0. \tag{27}$$

Similarly we define

$$H'(x,y)=\sum_{r,s,k}kH_{r,s,k}y^{2r}x^{2s}=\sum_m H'_m(x,y) \tag{28}$$

where

$$H'_{m+1}=x^2(H'_m+H_m)+y^2\sum_{n=0}^{\infty}[H'_{m+n+1}+(m+1)H_{m+n+1}]$$
$$+y^2\sum_{n=1}^{m}[G'_n+(m+1)G_n]x^{2(m+1-n)}. \tag{29}$$

A graphical representation of (29) is given in figure 4. It follows from (29) that

$$H'_{m+2}-(1+x^2-y^2)H'_{m+1}+x^2H'_m$$
$$=(m+2)H_{m+2}-(m+1)(1+x^2)H_{m+1}+mx^2H_m$$
$$+(1-y^2)G'_{m+2}-(1-y^2+x^2)G'_{m+1}+x^2G'_m-(m+2)G_{m+2}$$
$$+(m+1)(1+x^2)G_{m+1}-mx^2G_m. \tag{30}$$

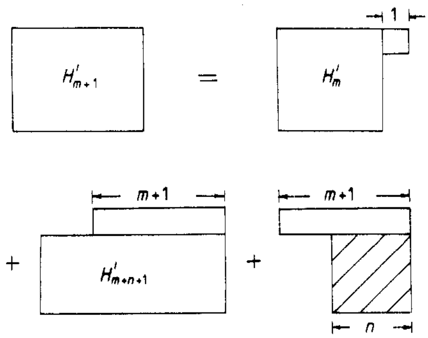

Figure 4. Graphical representation of equation (29).

The general solution of the recursion relation (30) is

$$H'_m = [A_+m(m-1)/2 + B_+m + C_+]z_+^m + [A_-m(m-1)/2 + B_-m + C_-]z_-^m + [Dm(m-1)/2 + Em + F]w^m \tag{31}$$

where

$$A_\pm = A(x, \pm y) \qquad B_\pm = B(x, \pm y) \qquad C_\pm = C(x, \pm y)$$

$$\begin{aligned} A(x, y) &= -y^3(1+y-x^2)/4[(1+y)^2 - x^2] \\ B(x, y) &= y^2(1+y)[(1+y)^2(3+y) - x^2(6+7y-y^2) + 3x^4]/8[(1+y)^2 - x^2]^2 \\ C(x, y) &= -y(1+y)[(1+y)^3 - x^2(2+2y+y^2-y^3) + x^4(1-y)]/8[(1+y)^2 - x^2]^2 \end{aligned} \tag{32}$$

$$D = -2x^2y^6/\Delta^{3/2}$$

$$E = -x^2y^4[2\Delta + y^2(1+x^2-y^2) + y^2\Delta^{1/2}]/\Delta^2.$$

It follows from (29) that

$$H'_2 - H'_1 = x^2(H'_1 + H_1) + x^2y^2(G'_1 + 2G_1 - 1) - y^2(H'_1 + H_1) + y^2 \sum_{n=2}^{\infty} H_n. \tag{33}$$

Substituting

$$\begin{aligned} H'_1 &= (B_+ + C_+)z_+ + (B_- + C_-)z_- + (E+F)w \\ H'_2 &= (A_+ + 2B_+ + C_+)z_+^2 + (A_- + 2B_- + C_-)z_-^2 + (D + 2E + F)w^2 \end{aligned} \tag{34}$$

into (33), we get

$$F = x^2y^4[1 - (x^2 - y^2)^2]/\Delta^2. \tag{35}$$

Summing over H'_m, we find

$$H' = x^2y^2[3 - 5(x^2+y^2) + (x^2+y^2)^2 + (x^2-y^2)(x^4-y^4)]/2\Delta^2 - x^2y^2[1-(x^2-y^2)^2]/2\Delta^{3/2}. \tag{36}$$

In the general case, a convex polygon can be divided uniquely into two (top and bottom) polygons as shown in figure 5. For the top polygon, the width at the bottom row equals the width of the corresponding bounding rectangle (the first special case turned upside down). The broken line is chosen such that the top polygon has the maximum area. The area k of the polygon is the summation of k_t (area of the top

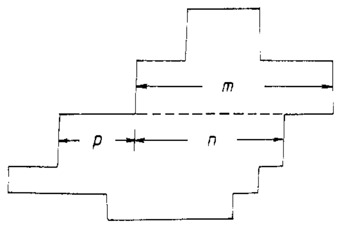

Figure 5. A convex polygon divided into two polygons by the broken line.

polygon) and k_b (area of the bottom polygon). Following the method of Lin and Chang (1988), it can be shown that

$$P_1(x, y) = \sum_{r,s,k} kP_{r,s,k}y^{2r}x^{2s} = P_t + P_b \tag{37}$$

where

$$P_t = \sum_{r,s,k} k_t P_{r,s,k}y^{2r}x^{2s}$$

$$= G' + 2\sum_{m=2}^{\infty} G'_m \sum_{n=1}^{m-1} x^{-2n} \sum_{p=0}^{\infty} H_{n+p} + \sum_{m=3}^{\infty} G'_m \sum_{n=1}^{m-2} G_n x^{-2n}(m-n-1) \tag{38}$$

$$P_b = \sum_{r,s,k} k_b P_{r,s,k}y^{2r}x^{2s}$$

$$= 2\sum_{m=2}^{\infty} G_m \sum_{n=1}^{m-1} x^{-2n} \sum_{p=0}^{\infty} H'_{n+p} + \sum_{m=3}^{\infty} G_m \sum_{n=1}^{m-2} G'_n x^{-2n}(m-n-1). \tag{39}$$

After a lengthy calculation, the final result is

$$P_1(x, y) = x^2y^2T/\Delta^4 + 4x^4y^4[1-(x^2-y^2)^2]/\Delta^{5/2} \tag{40}$$

where

$$\begin{aligned} T = 1 &- 6(x^2+y^2) + 15(x^4+y^4) + 20x^2y^2 - 20(x^6+y^6) \\ &- 18(x^4y^2+x^2y^4) + 15(x^8+y^8) - 8(x^6y^2+x^2y^6) + 28x^4y^4 - 6(x^{10}+y^{10}) \\ &+ 22(x^8y^2+x^2y^8) - 40(x^6y^4+x^4y^6) + x^{12} + y^{12} - 12(x^{10}y^2+x^2y^{10}) \\ &- 5(x^8y^4+x^4y^8) + 64x^6y^6 + 2(x^{12}y^2+x^2y^{12}) \\ &+ 18(x^{10}y^4+x^4y^{10}) - 20(x^8y^6+x^6y^8) + 2(x^{12}y^4+x^4y^{12}) \\ &- 8(x^{10}y^6+x^6y^{10}) + 12x^8y^8. \end{aligned}$$

When $x = y$, equation (40) agrees with (6), which was obtained non-rigorously by Enting and Guttmann (1989).

The second conjectured result of Enting and Guttmann is much harder to prove. In principle our method can be generalised to derive the second and higher moments of the area of convex polygons. Unfortunately the labour required increases rapidly. It is possible that there exists a simpler way to handle this problem.

Acknowledgments

I thank Professor Ira Gessel of Brandeis University for calling my attention to the interesting papers by Temperley, Kim, Delest and Viennot. I also thank Dr Enting and Professor Guttmann for sending their preprint to the library of Tsing Hua University. This research is supported by the National Science Council of the Republic of China.

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